JOURNAL OF APPROXIMATION THEORY 78, 314–320 (1994)

Pointwise Best Approximation in the Space of Strongly Measurable Functions with Applications to Best Approximation in $L^p(\mu, X)$

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Communicated by Will Light

Received January 23, 1992; accepted in revised form June 28, 1993

The object of this paper is to prove the following theorem: Let Y be a closed subspace of the Banach space X, (S, Σ, μ) a σ-finite measure space, $L(S, Y)$ (respectively, $L(S, X)$) the space of all strongly measurable functions from S to Y (respectively, X), and p a positive number. Then $L(S, Y)$ is pointwise proximinal in $L(S, X)$ if and only if $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$. As an application of the theorem stated above, we prove that if Y is a separable closed subspace of the Banach space X, p is a positive number, then $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ if and only if Y is proximinal in X. Finally, several other interesting results on pointwise best approximation are also obtained. © 1994 Academic Press, Inc.

1. INTRODUCTION

Recently, best approximation in $L^p(\mu, X)$ was discussed deeply by Khalil and Deeb in [1]. The central purpose of the present paper is to study further the topics stated above starting from a new angle. Precisely, we establish an equivalent relation between the proximinality of $L^p(\mu, Y)$ in $L^p(\mu, X)$ and the pointwise proximinality of $L(S, Y)$ in $L(S, X)$.

It should be pointed out that the results obtained in the present paper sharpen and improve those in [1, 2], and our methods are not only distinct from those in [1] but also seem to be more unified and simpler than those in [1].

In Section 2, we introduce some basic concepts and known results for the reader's convenience. In section 3, we give and prove our main results.

0021-9045/94 $6.00

Throughout the whole paper, we always suppose $(X, \|\cdot\|)$ is a Banach space over the number field K, Y is a closed subspace of X, K is the field R of real numbers or the field C of complex numbers, (S, Σ, μ) is a given σ-finite measure space, $L(S, K)$ is the ring of all K-valued measurable functions defined on(S, Σ, μ) under pointwise addition and multiplication, in which functions equal a.e. are identified. For $A \subset \Omega$, we write I_A for the indicator function of A, A' for the complement of A, i.e., $A' = \Omega \setminus A$. Finally, N stands for the set of all natural numbers.

2. Preliminaries

Definition 2.1 [6]. Let (M, d) be a metric space, 2^M the collection of all nonempty subsets of M. A multifunction F: $S \to 2^M$ is called compactly measurable if $F^{-1}(B) = \{s \in S | F(s) \cap B \neq \emptyset\} \in \Sigma$ for any compact subset B of M.

Lemma 2.1 [6, Th. 1', p. 230]. *Let (M, d) be a separable metric space. Then a multifunction F: $S \to 2^M$ having complete values, namely, that $F(s)$ is a complete subset of M for each $s \in S$, is compactly measurable if and only if there exists a countable set $\{V_n | n \in N\}$ of M-valued Borel measurable functions defined on (S, Σ, μ) such that $F(s) = \overline{\{V_n(s) | n \in N\}}$ (the closure of the set $\{V_n(s) | n \in N\}$) for each $s \in S$.*

Definition 2.2 [3, 4, 7, 8]. Let (M, d) be a metric space. A Borel measurable function from S to M is called strongly measurable if it is the pointwise limit of a sequence of simple Borel measurable functions from S to M. It is clear that the notation of "strongly measurable" coincides with that of "Borel measurable" if M is separable.

Denote by $L(S, X)$ the space of all strongly measurable functions from (S, Σ, μ) to $(X, \|\cdot\|)$. We say that a subset M of $L(S, X)$ is closed if it is closed with respect to pointwise limits of sequences. Functions equal a.e. are identified. Let $M \subset L(S, X)$ be a closed $L(S, K)$-submodule (for example, $L(S, Y)$), for some $r > 0$, we write $L^r(M)$ for the Banach space (when $r \geq 1$) with the norm $\|\cdot\|_r$ (the Frechet space (when $0 < r < 1$) with the quasinorm $\|\cdot\|_r$) of all functions p in M satisfying $\int_S \|p(s)\|^r \, d\mu < +\infty$, it is clear that $L^r(M) = L^r(\mu, Y)$ when $M = L(S, Y)$.

Remark 2.1. When X is not separable, examples show that the set of all Borel measurable functions from S to X does not form a linear space (see [9]).

Definition 2.3. Let $f \in L(S, X)$, $D \subset L(S, X)$. $f_0 \in D$ is called a pointwise best approximant of f in D if for all $g \in D$, we have $\|f(s) - f_0(s)\| \leq \|f(s) - g(s)\|$ a.e. "Pointwise proximinal" is defined accordingly.

Throughout the rest of the paper, such concepts as "Proximinal" and "Best approximation" for $(L^r(M), \|\cdot\|_r)$ are the same as those in [1].

LEMMA 2.2. *Let M be a closed subspace of $L(S, X)$. If M is pointwise proximinal in $L(S, X)$, then M must be an $L(S, K)$-submodule.*

Proof. The set of all simple measurable functions in $L(S, K)$ is dense in $L(S, K)$, and M is a closed linear subspace of $L(S, X)$. Thus to prove M is an $L(S, K)$-submodule, we only need to prove $I_A \cdot p \in M$ for any $A \in \Sigma$, $p \in M$.

Suppose otherwise, then there exist $A \in \Sigma$ and $p \in M$ such $p_A = I_A \cdot p \overline{\in} M$. Since M is pointwise proximinal in $L(S, X)$, there is $p'_A \in M$ which is a pointwise best approximant of p_A in M. Since $\theta \in M$, we have $\|p_A(s) - p'_A(s)\| \leq \|p_A(s) - \theta(s)\| = I_A(s) \cdot \|p(s)\|$ a.e. This implies $I_{A'}(s) \cdot \|p_A(s) - p'_A(s)\| = 0$ a.e. However, $p \in M$ implies $\|p_A(s) - p'_A(s)\| \leq \|p_A(s) - p(s)\| = I_{A'}(s) \cdot \|p(s)\|$ a.e., and hence $I_A(s) \cdot \|p_A(s) - p'_A(s)\| = 0$ a.e. Consequently $\|p_A(s) - p'_A(s)\| = 0$ a.e. This implies $p_A = p'_A \in M$. The contradiction ends the proof of Lemma 2.2.

LEMMA 2.3. *Let M be a closed $L(S, K)$-submodule of $L(S, X)$, $r > 0$, and $p \in L^r(\mu, X)$. If $p_0 \in L^r(M)$ is a best approximant of p in $L^r(M)$, then p_0 is also a pointwise best approximant of p in M.*

Proof. Since $L^r(M)$ is sequentially dense in M under pointwise convergence, to prove our desired result, we only need to prove $\|p(s) - p_0(s)\| \leq \|p(s) - q(s)\|$ a.e. for any $q \in L^r(M)$.

Suppose otherwise, i.e., there exists some $q_0 \in L^r(M)$ such that $\mu\{s \in S \mid \|p(s) - p_0(s)\| > \|p(s) - q_0(s)\|\} > 0$. Let $A = \{s \in S \mid \|p(s) - p_0(s)\| > \|p(s) - q_0(s)\|\}$ and $p'_0 = I_A \cdot q_0 + I_{A'} \cdot p_0$. From the fact that M is an $L(S, K)$-submodule, we see that $p'_0 \in M$. It is clear that $p'_0 \in L^r(M)$ and $\|p - p'_0\|_r < \|p - p_0\|_r$, this contradiction with the hypothesis on p_0 ends the proof of Lemma 2.3.

3. MAIN RESULTS

THEOREM 3.1. *Let M be a closed $L(S, K)$-submodule of $L(S, X)$, $r > 0$. Then the following are equivalent:*

(1) *M is pointwise proximinal in $L(S, X)$;*

(2) *$(L^r(M), \|\cdot\|_r)$ is proximinal in $(L^r(\mu, X), \|\cdot\|_r)$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear, we prove (2) $\Rightarrow$ (1) as follows.

Since (S, Σ, μ) is σ-finite, we can suppose $S = \bigcup_{n \in N} A_n$, $A_n \in \Sigma$ and $A_n \subset A_{n+1}$ such that $\mu(A_n) < +\infty$ for each $n \in N$.

For any given $p \in L(S, X)$, put $B_n = \{s \in S \mid \|p(s)\| \le n\}$. Then we have $\mu(S \setminus \bigcup_{n \in N} B_n) = 0$. Set $C_n = A_n \cap B_n$, it is clear that $\mu(S \setminus \bigcup_{n \in N} C_n) = 0$.

Put $D_n = C_n \setminus C_{n-1}$ and $p_n = I_{D_n} \cdot p$. Then $p_n \in L^r(\mu, X)$, and hence there exists $p'_n \in L^r(M)$ such that $\|p_n - p'_n\|_r = \inf_{q \in L^r(M)} \|p_n - q\|_r$. From Lemma 2.3 we have the following:

$$\|p_n(s) - p'_n(s)\| \le \|p_n(s) - q(s)\| \qquad \text{a.e. for any } q \in M. \quad (3.1.1)$$

Taking $q = 0$ yields $\|p'_n(s)\| \le 2\|p_n(s)\| = 2I_{D_n}(s) \cdot \|p(s)\|$ a.e., and thus $I_{D'_n}(s) \cdot \|p'_n(s)\| = 0$ a.e. (3.1.2)

Particularly, (3.1.2) implies $p'_n = I_{D_n} \cdot p'_n$ for each $n \in N$. (3.1.3)

By setting $p^0_n = \sum_{k=1}^{n} p'_k$ for each $n \in N$, we get $p^0_{n+k} - p^0_n = \sum_{m=n+1}^{n+k} p'_m$ for any n and k in N, and thus we have the following:

$$\|p^0_{n+k}(s) - p^0_n(s)\| \le \sum_{m=n+1}^{n+k} \|p'_m(s)\| \text{ a.e.} \qquad (3.1.4)$$

Since $\mu(S \setminus \bigcup_{n \in N} D_n) = 0$ and $\|p^0_{n+k}(s) - p^0_n(s)\| \le \sum_{m=n+1}^{n+k} \|p'_m(s)\| = 2(\sum_{m=n+1}^{n+k} I_{D_m}(s))\|p(s)\|$ a.e. (note (3.1.2) and (3.1.4)), and hence $I_{C_n}(s) \cdot \|p^0_{n+k}(s) - p^0_n(s)\| = 0$ a.e. This implies $\lim_{n \to \infty} \|p^0_{n+k}(s) - p^0_n(s)\| = 0$ a.e. uniformly for all $k \in N$; that is, $\{p^0_n | n \in N\}$ is a Cauchy sequence.

Since M is closed, and hence also complete, there is $p_0 \in M$ such that $\{p^0_n\}$ converges to p_0. We can write $p_0 = \sum_{k=1}^{\infty} p'_k$. From (3.1.3) we have $p_0 = \sum_{k=1}^{\infty} I_{D_k} \cdot p'_k$. Now we assert

$$\|p(s) - p_0(s)\| \le \|p(s) - q(s)\| \qquad \text{a.e. for any } q \in M. \quad (3.1.5)$$

If (3.1.5) is false, then there exists some $q_0 \in M$ such that $\mu\{s \in S \mid \|p(s) - p_0(s)\| > \|p(s) - q_0(s)\|\} > 0$. Let $A_0 = \{s \in S \mid \|p(s) - p_0(s)\| > \|p(s) - q_0(s)\|\}$. Since $\mu(S \setminus \bigcup_{n=1}^{\infty} D_n) = 0$, there is at least some $k_0 \in N$ such that $\mu(A_0 \cap D_{k_0}) > 0$. From the fact that $\|p_{k_0}(s) - p^0_{k_0}(s)\| = I_{D_{k_0}}(s) \cdot \|p(s) - p_0(s)\|$ a.e. and $\|p_{k_0}(s) - q^0_{k_0}(s)\| = I_{D_{k_0}}(s) \cdot \|p(s) - q_0(s)\|$ a.e., where $p^0_{k_0} = I_{D_{k_0}} \cdot p_0$ and $q^0_{k_0} = I_{D_{k_0}} \cdot q_0$, we have $\{s \in S \mid \|p_{k_0}(s) - p^0_{k_0}(s)\| > \|p_{k_0}(s) - q^0_{k_0}(s)\|\} \supset A_0 \cap D_{k_0}$ a.e., so $\mu\{s \in S \mid \|p_{k_0}(s) - p^0_{k_0}(s)\| > \|p_{k_0}(s) - q^0_{k_0}(s)\|\} \ge \mu(A_0 \cap D_{k_0}) > 0$. (3.1.6)

Since $p^0_{k_0} \in M$, $q^0_{k_0} \in M$, (3.1.6) contradicts (3.1.1). So p_0 is a pointwise best approximant of p in M.

This completes the proof of Theorem 3.1.

THEOREM 3.2. *Let $p > 0$. Then the following are equivalent*:

(1) *$L(S, Y)$ is pointwise proximinal in $L(S, X)$*;

(2) *$L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$*.

Proof. In Theorem 3.1, we take $M = L(S, Y)$, then the proof of Theorem 3.2 immediately follows from Theorem 3.1.

COROLLARY 3.1. *Let $p > 1$. Then the following are equivalent*:

(1) *$L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$*;

(2) *$L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$*.

Proof. Both (1) and (2) are equivalent with the fact $L(S, Y)$ is pointwise proximinal in $L(S, X)$, and hence (1) and (2) are equivalent.

This completes the proof of Corollary 3.1.

Remark 3.1. When (S, Σ, μ) is a finite measure space, Corollary 3.1 is just Thm. 1.1 of [1].

THEOREM 3.3. *If $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$, then Y must be proximinal in X.*

Proof. We take $\{A_n\}_{n=1}^{\infty}$ to be the same as Theorem 3.1, then there must be $k_0 \in N$ such that $0 < \mu(A_{k_0} < +\infty$.

Let x be any fixed point of X. Define p: $S \to X$ by $p(s) = I_{A_{k_0}}(s) \cdot x$ for each $s \in S$, then clearly $p \in L^1(\mu, X)$, and hence there is at least some $p_0 \in L^1(\mu, Y)$ such that $\|p - p_0\|_1 = \inf_{q \in L^1(\mu, Y)} \|p - q\|_1$. So $\|p - p_0\|_1 \leq \|p - I_{A_{k_0}} \cdot y\|_1 = \mu(A_{k_0})\|x - y\|$ for any $y \in Y$, i.e.,

$$\int_S \left\| I_{A_{k_0}}(s)x - p_0(s) \right\| du(s) \leq \mu(A_{k_0})\|x - y\| \quad \text{for each } y \in Y.$$

Put $x_0 = \int_S p_0(s)\, du(s)$, then

$$\left\| x - \frac{1}{\mu(A_{k_0})} x_0 \right\| \leq \frac{1}{\mu(A_{k_0})} \int_S \|p - p_0\|\, du \leq \|x - y\|$$

for any $y \in Y$. From $(1/(\mu(A_{k_0})))x_0 \in Y$, we see that

$$\left\| x - \frac{1}{\mu(A_{k_0})} x_0 \right\| = \inf_{y' \in Y} \|x - y\|,$$

that is, $(1/(\mu(A_{k_0})))x_0$ is a best approximant of x in Y.

This completes the proof of Theorem 3.3.

THEOREM 3.4. *Let Y be a separable and proximinal subspace of X. Then $L(S,Y)$ is pointwise proximinal in $L(S,X)$.*

Proof. Let $p \in L(S,X)$. Now we define a multifunction F: $(S,\Sigma,\mu) \to 2^Y$ as follows: $F(s) = \{y \in Y |\ \|p(s) - y\| = \inf_{y' \in Y}\|p(s) - y'\|\}$ for each $s \in S$. Then $F(s)$ is nonempty and closed, and hence also complete. Now we prove F is compactly measurable:

Let B be any compact subset of Y. Then we see the following relations:

$$\begin{aligned} F^{-1}(B) &= \{s \in S | F(s) \cap B \neq \varnothing\} \\ &= \bigcup_{y \in B} \left\{ s \in S |\ \|p(s) - y\| = \inf_{y' \in Y} \|p(s) - y'\| \right\} \\ &= \bigcap_{j=1}^{\infty} \bigcup_{i \in N} \left\{ s \in S |\ \|p(s) - y_i\| \leq \inf_{y' \in Y} \|p(s) - y'\| + \frac{1}{j} \right\}, \end{aligned}$$

where $\{y_i | i \in N\}$ is any selected countable subset of B which is dense in B.

Since the function f: $X \to [0, +\infty]$ defined by $f(x) = \inf_{y \in Y}\|x - y\|$ is continuous and p: $S \to X$ is a strongly measurable function, $f(p(\cdot))$: $S \to R^+$ is a real-valued measurable function defined on S, and hence $F^{-1}(B) \in \Sigma$; namely, F is compactly measurable.

From Lemma 2.1, we see that there exists a sequence $\{V_n\}$ in $L(S,Y)$ such that $F(s) = \overline{\{V_n(s) | n \in N\}}$ for each $s \in S$.

According to the definition of F, it is easily seen that each V_n is a pointwise best approximant of p in $L(S,Y)$.

This completes the proof of Theorem 3.4.

THEOREM 3.5. *Let Y be separable, $p > 0$. Then $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ if and only if Y is proximinal in X.*

Proof. The proof follows immediately from Theorem 3.2, Theorem 3.3, and Theorem 3.4.

Remark 3.2. Theorem 3.5 removes an extra condition in Thm. 1.3 of [1], namely, that Y should be also a dual space.

THEOREM 3.6. *Let Y be reflexive. Then $L(S,Y)$ is pointwise proximinal in $L(S,X)$.*

Proof. Since $L^2(\mu, Y)$ is reflexive, it is proximinal in $L^2(\mu, X)$, and hence $L(S,Y)$ is also pointwise proximinal in $L(S,X)$ from Theorem 3.2.

This completes the proof of Theorem 3.6.

THEOREM 3.7. *X is reflexive if and only if any closed $L(S,K)$-submodule of $L(S,X)$ is pointwise proximinal in $L(S,X)$.*

Proof. Necessity. Let M be a closed $L(S, K)$-submodule of $L(S, X)$. Then $L^2(M)$ is a closed subspace of $L^2(\mu, X)$. Since X is reflexive, so is $L^2(\mu, X)$ (see [5]), and hence $L^2(M)$ is proximinal in $L^2(\mu, X)$. M is pointwise proximinal in $L(S, X)$ from Theorem 3.1.

Sufficiency. Let Y be a closed subspace of X. Then $L(S, Y)$ is a closed $L(S, K)$-submodule of $L(S, X)$, and hence $L(S, Y)$ is pointwise proximinal in $L(S, X)$, so Y is also proximinal in X from Theorem 3.3. It is well known that X is reflexive iff any closed subspace of X is proximinal in X from [5], and thus X is a reflexive Banach space.

This completes the proof of Theorem 3.7.

THEOREM 3.8. *Let X be a Hilbert space, $M \subset L(S, X)$ a closed subspace. Then M is pointwise proximinal in $L(S, X)$ iff M is an $L(S, K)$-submodule.*

Proof. The proof of Theorem 3.8 follows from Lemma 2.2 and Theorem 3.7.

ACKNOWLEDGMENT

The authors thank the referee for his valuable suggestion, which considerably improves the readability of this paper.

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